常用数学公式

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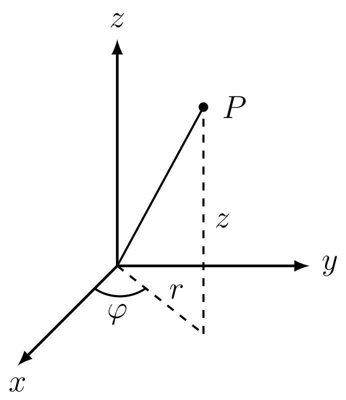
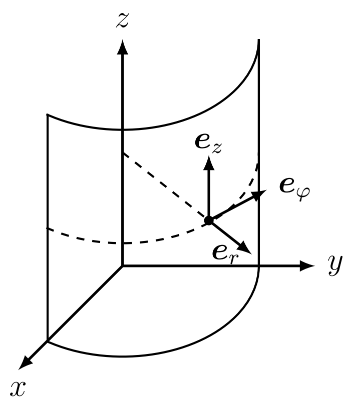
1 曲线坐标系

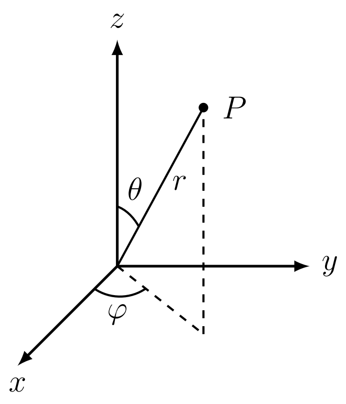
1.1 变量定义与基矢方向

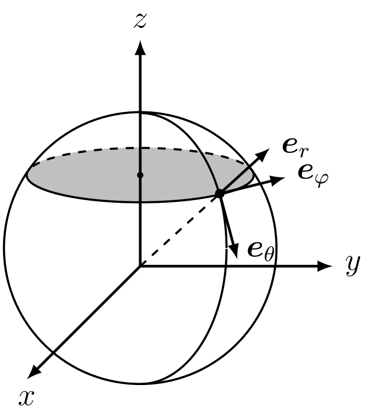
柱坐标系





球坐标系





1.2 有向线元与体元

对于一般的曲线坐标系来说，有向线元和体元的表达式为

$$\mathrm{d}\boldsymbol{r} = h_1 \mathrm{d}u_1 \boldsymbol{e}_1 + h_2 \mathrm{d}u_2 \boldsymbol{e}_2 + h_3 \mathrm{d}u_3 \boldsymbol{e}_3$$

$$\mathrm{d}V = h_1 h_2 h_3 \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3$$

在直角坐标系中

$$u_1 = x\ , u_2 = y\ , u_3 = z$$

$$h_1 = 1\ , h_2 = 1\ , h_3 = 1$$

在柱坐标系中

$$u_1 = r\ , u_2 = \varphi\ , u_3 = z$$

$$h_1 = 1\ , h_2 = r\ , h_3 = 1$$

在球坐标系中

$$u_1 = r\ , u_2 = \theta\ , u_3 = \varphi$$

$$h_1 = 1\ , h_2 = r\ , h_3 = r\sin\theta$$

2 高斯公式和斯托克斯公式

高斯公式

$$\oiint_{\partial\Omega} \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{S} = \iiint_{\Omega} (\nabla \cdot \boldsymbol{f}) \mathrm{d}V$$

斯托克斯公式

$$\oint_{\partial\Sigma} \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{l} = \iint_{\Sigma} (\nabla \times \boldsymbol{f}) \cdot \mathrm{d}\boldsymbol{S}$$

3 梯度、散度、旋度与拉普拉斯算符

3.1 通式

梯度

$$\nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\boldsymbol{e}_3$$

散度

$$\nabla \cdot \boldsymbol{f} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3f_1) + \frac{\partial}{\partial u_2}(h_3h_1f_2) + \frac{\partial}{\partial u_3}(h_1h_2f_3)\right]$$

旋度

$$\nabla \times \boldsymbol{f} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1f_1 & h_2f_2 & h_3f_3 \end{vmatrix}$$

拉普拉斯算符

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

3.2 柱坐标系

$$\nabla\psi=\frac{\partial\psi}{\partial r}\boldsymbol{e}_r+\frac{1}{r}\frac{\partial\psi}{\partial\varphi}\boldsymbol{e}_\varphi+\frac{\partial\psi}{\partial z}\boldsymbol{e}_z$$

$$\nabla\cdot\boldsymbol{f}=\frac{1}{r}\frac{\partial}{\partial r}(rf_r)+\frac{1}{r}\frac{\partial f_\varphi}{\partial\varphi}+\frac{\partial f_z}{\partial z}$$

$$\nabla\times\boldsymbol{f}=\left(\frac{1}{r}\frac{\partial f_z}{\partial\varphi}-\frac{\partial f_\varphi}{\partial z}\right)\boldsymbol{e}_r+\left(\frac{\partial f_r}{\partial z}-\frac{\partial f_z}{\partial r}\right)\boldsymbol{e}_\varphi+\left[\frac{1}{r}\frac{\partial}{\partial r}(rf_\varphi)-\frac{1}{r}\frac{\partial f_r}{\partial\varphi}\right]\boldsymbol{e}_z$$

$$\nabla^2\psi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2}+\frac{\partial^2\psi}{\partial z^2}$$

3.3 球坐标系

$$\nabla\psi=\frac{\partial\psi}{\partial r}\boldsymbol{e}_r+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\boldsymbol{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\boldsymbol{e}_\varphi$$

$$\nabla\cdot\boldsymbol{f}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2f_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta)+\frac{1}{r\sin\theta}\frac{\partial f_\varphi}{\partial\varphi}$$

$$\nabla\times\boldsymbol{f}=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta f_\varphi)-\frac{\partial f_\theta}{\partial\varphi}\right]\boldsymbol{e}_r+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial f_r}{\partial\varphi}-\frac{\partial}{\partial r}(rf_\varphi)\right]\boldsymbol{e}_\theta+\frac{1}{r}\left[\frac{\partial}{\partial r}(rf_\theta)-\frac{\partial f_r}{\partial\theta}\right]\boldsymbol{e}_\varphi$$

$$\nabla^2\psi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}$$

4 矢量分析

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$$

$$\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})=(\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a}-(\boldsymbol{c}\cdot\boldsymbol{a})\boldsymbol{b}$$

$$\nabla(\varphi\psi)=\varphi\nabla\psi+\psi\nabla\varphi$$

$$\nabla\cdot(\varphi\boldsymbol{f})=(\nabla\varphi)\cdot\boldsymbol{f}+\varphi\nabla\cdot\boldsymbol{f}$$

$$\nabla\times(\varphi\boldsymbol{f})=(\nabla\varphi)\times\boldsymbol{f}+\varphi\nabla\times\boldsymbol{f}$$

$$\nabla(\boldsymbol{f}\cdot\boldsymbol{g})=\boldsymbol{f}\times(\nabla\times\boldsymbol{g})+(\boldsymbol{f}\cdot\nabla)\boldsymbol{g}+\boldsymbol{g}\times(\nabla\times\boldsymbol{f})+(\boldsymbol{g}\cdot\nabla)\boldsymbol{f}$$

$$\nabla\cdot(\boldsymbol{f}\times\boldsymbol{g})=(\nabla\times\boldsymbol{f})\cdot\boldsymbol{g}-\boldsymbol{f}\cdot(\nabla\times\boldsymbol{g})$$

$$\nabla\times(\boldsymbol{f}\times\boldsymbol{g})=(\boldsymbol{g}\cdot\nabla)\boldsymbol{f}+(\nabla\cdot\boldsymbol{g})\boldsymbol{f}-(\boldsymbol{f}\cdot\nabla)\boldsymbol{g}-(\nabla\cdot\boldsymbol{f})\boldsymbol{g}$$

$$\nabla \cdot \nabla \varphi = \nabla^2 \varphi$$

$$\nabla \times \nabla \varphi = 0$$

$$\nabla \cdot (\nabla \times \boldsymbol{f}) = 0$$

$$\nabla \times (\nabla \times \boldsymbol{f}) = \nabla (\nabla \cdot \boldsymbol{f}) - \nabla^2 \boldsymbol{f}$$

5 傅里叶级数

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}$$

其中

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \mathrm{d}x$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \mathrm{d}x$$

6 积分

$$\int_{-l}^{l} \cos^2 \frac{n\pi x}{l} \mathrm{d}x = l$$

$$\int_{-l}^{l} \sin^2 \frac{n\pi x}{l} \mathrm{d}x = l$$

$$\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}$$

$$\int_{0}^{\infty} e^{-ax^2} \mathrm{d}x = \frac{1}{2} \sqrt{\frac{\pi}{a}} \quad (a > 0)$$

$$\int_{0}^{\infty} x^n e^{-ax} \mathrm{d}x = \frac{n!}{a^{n+1}}$$

$$\int_0^{\pi} \sin nx \mathrm{d}x = \frac{1 - (-1)^n}{n}$$

$$\int_0^{\pi} x \sin nx \mathrm{d}x = -\frac{(-1)^n \pi}{n}$$

$$\int_0^{\pi} x^2 \sin nx \mathrm{d}x = \frac{2[(-1)^n - 1]}{n^3} - \frac{(-1)^n \pi^2}{n}$$

$$\int_0^{\pi} \cos nx \mathrm{d}x = 0$$

$$\int_0^{\pi} x \cos nx \mathrm{d}x = \frac{(-1)^n - 1}{n^2}$$

$$\int_0^{\pi} x^2 \cos nx \mathrm{d}x = \frac{(-1)^n 2\pi}{n^2}$$

7 复级数

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty)$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad (|z| < \infty)$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \quad (|z| < \infty)$$

8 三角函数

8.1 积化和差

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin\left(\alpha+\beta\right)+\sin\left(\alpha-\beta\right)\right]$$
$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin\left(\alpha+\beta\right)-\sin\left(\alpha-\beta\right)\right]$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos\left(\alpha+\beta\right)+\cos\left(\alpha-\beta\right)\right]$$
$$\sin\alpha\sin\beta = -\frac{1}{2}\left[\cos\left(\alpha+\beta\right)-\cos\left(\alpha-\beta\right)\right]$$

8.2 和差化积

$$\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha-\sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha-\cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

8.3 和差角公式

$$\sin\left(\alpha+\beta\right) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin\left(\alpha-\beta\right) = \sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\cos\left(\alpha+\beta\right) = \cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\cos\left(\alpha-\beta\right) = \cos\alpha\cos\beta+\sin\alpha\sin\beta$$

9 其他

在球坐标系中

$$\nabla\cdot\left(\frac{1}{r^2}\boldsymbol{e}_r\right) = 0$$
$$\nabla\frac{1}{r} = -\frac{1}{r^2}\boldsymbol{e}_r$$
$$\nabla^2\frac{1}{r} = -4\pi\delta(\boldsymbol{r})$$